Differential equations for general
$SU(n)$ Bethe ansatz systems

Patrick Dorey[1], Clare Dunning[2] and Roberto Tateo[3]

[1]*SPhT Saclay, 91191 Gif-sur-Yvette cedex, France*

[1,2]*Dept. of Mathematical Sciences, University of Durham, Durham DH1 3LE, UK*

[3]*Universiteit van Amsterdam, Inst. voor Theoretische Fysica, 1018 XE Amsterdam, NL*

Abstract

We show that $SU(n)$ Bethe Ansatz equations with arbitrary 'twist' parameters are hidden inside certain n^{th} order ordinary differential equations, and discuss various consequences of this fact.

[1]`p.e.dorey@durham.ac.uk`
[2]`t.c.dunning@durham.ac.uk`
[3]`tateo@wins.uva.nl`

1 Introduction

Recent work has revealed unexpected connections between the functional relations and Bethe ansatz (BA) equations arising in integrable quantum field theory, and the behaviour of certain ordinary differential equations (ODEs) [1–7]. In this paper we extend these links to cover a whole class of BA systems associated with the $SU(n)$ Lie algebras. The papers just cited, and earlier work such as [8–11], should be consulted for the background to these developments, but we begin with a brief review of the main points.

The first example to be found concerned $SU(2)$ BA systems at a specific value of the twist parameter, and associated them with certain Schrödinger equations [1]. The generalisation to arbitrary twist turned out to require the addition of an angular-momentum term to the ODE [2]. In studies of the Bethe ansatz, a functional equation found by Baxter, the T-Q relation, is known to encode the BA equations in a particularly neat way; the place of this relation in the differential equations side of the story was found in [4]. After this, the question of generalisations to $SU(n)$ with $n > 2$ was addressed in [5,6]. In [5], a reasonably complete mapping of the $SU(3)$ case onto third-order ODEs was found, and it was suggested that higher-order equations would be the appropriate setting to search for the more general case. Independent work by Suzuki [6] provided some more concrete support for this idea. However, this latter paper did not treat the completely general case (both the deformation parameter and the twists were restricted) and the role of a particularly important generalisation of the T-Q relation, sometimes called the 'dressed vacuum form', was not elucidated. For this reason, we return to the topic here, and show how the approach of [5] can be extended to general n. One of the main new features is a way of incorporating angular-momentum type terms into the higher-order ODEs which encodes the BA twists in a particularly neat way. This simplifies the discussion considerably, and even streamlines aspects of the $SU(3)$ case treated in [5]. We also derive nonlinear integral equations (NLIEs) for the associated spectral problems, and discuss duality properties.

2 The ordinary differential equation

The initial ODE that one might consider has the form

$$\left((-1)^{n+1} \frac{d^n}{dx^n} + P(x, E) \right) \psi(x) = 0 \,, \qquad P(x, E) = x^{nM} - E \qquad (2.1)$$

with M a positive real number, related to the deformation parameter. For $n=3$ this is the first of the equations studied in [5], while the general-n case, but with nM restricted to integer values, was the subject of [6]. For each n, (2.1) has only one free parameter, namely M, and so it is clear that this equation cannot hope to incorporate $SU(n)$ BA systems at general values of the twists. The results of [2,5] would tend to suggest that these twists should be associated with the addition of terms homogeneous with $\frac{d^n}{dx^n}$, of the form $A_k x^{-k} \frac{d^{n-k}}{dx^{n-k}}$, $k = 2 \ldots n$. (It will be convenient to assume that the term with

$k = 1$ has been eliminated by a suitable redefinition of ψ.) This gives $n-1$ further free parameters, exactly matching the number of twists in an $SU(n)$ BA system. However, the variables $\{A_k\}$ turn out to be rather inconvenient, and it is better to take a slightly more indirect route.

First, define a general homogeneous differential operator of degree one by setting

$$D(g) = \left(\frac{d}{dx} - \frac{g}{x} \right) . \tag{2.2}$$

Useful properties of this operator are

$$D(g)^\dagger = -D(-g) , \tag{2.3}$$

$$D(g_2 - 1)D(g_1) = D(g_1 - 1)D(g_2) . \tag{2.4}$$

The first of these relates the operator to its adjoint, while the second expresses a form of commutativity. Now, given a vector $\mathbf{g} = (g_0, g_1, \dots g_{n-1})$, set

$$D(\mathbf{g}) = D(g_{n-1} - (n{-}1)) \, D(g_{n-2} - (n{-}2)) \, \dots \, D(g_1 - 1) \, D(g_0) . \tag{2.5}$$

This is a homogeneous differential operator of order n; by the 'commutativity' property it depends on the components $\{g_0 \dots g_{n-1}\}$ of $\mathbf{g}$ in a symmetrical manner. From now on we also impose

$$\sum_{i=0}^{n-1} g_i = \frac{n(n{-}1)}{2} \tag{2.6}$$

to ensure the vanishing of the term in $D(\mathbf{g})$ proportional to $x^{-1} \frac{d^{n-1}}{dx^{n-1}}$. Finally, we record one more property of $D(\mathbf{g})$ which will be useful later: its indicial polynomial (see for example [12]) is

$$f(\lambda) = \prod_{i=0}^{n-1} (\lambda - g_i) . \tag{2.7}$$

With these ingredients in place, the nth order ODE that we propose to study is obtained simply by replacing the operator $\frac{d^n}{dx^n}$ in (2.1) by $D(\mathbf{g})$:

$$\left((-1)^{n+1} D(\mathbf{g}) + P(x, E) \right) \psi(x) = 0 , \qquad P(x, E) = x^{nM} - E . \tag{2.8}$$

3 The fundamental system of solutions

As in [4–7] we take our cue from the approach of [8] to second-order ODEs. We claim that (2.8) has a solution $y(x, E, \mathbf{g})$ such that:
• $y(x, E, \mathbf{g})$ is an entire function of $(x, E, \mathbf{g})$, modulo a possible branch point at the origin of the complex x plane.

- as $|x| \to \infty$ in the sector $|\arg x| < (n+1)\pi/n(M+1)$,

$$\frac{d^p y}{dx^p} \sim (-1)^p \frac{x^{(1-n+2p)M/2}}{i^{(n-1)/2}\sqrt{n}} \exp(-x^{M+1}/(M+1)) \tag{3.1}$$

for $p = 0, 1, \ldots$. This holds if $M > 1/(n-1)$; more generally the WKB-like formula $y \sim P(x,E)^{-(n-1)/2n} \exp(-\int^x P(t,E)^{1/n} dt)$ can be used in the way explained, for $n=2$, in the appendix of [4]. The normalisation of y has been chosen for later convenience.
- y is uniquely characterised by the asymptotic (3.1) for $p = 0$ and x real.

For $n=2$, $nM \in \mathbb{Z}$, and $D(\mathbf{g})=\frac{d^2}{dx^2}$, these properties were proved in [8]. We have not attempted a proof for the more general case; however, see [6] for a related discussion.

To construct further solutions, we set*

$$y_k(x, E, \mathbf{g}) = \omega^{(n-1)k/2} \, y(\omega^{-k} x, \omega^{-nMk} E, \mathbf{g}) \ , \qquad \omega = e^{2\pi i/n(M+1)} . \tag{3.2}$$

For $k \in \mathbb{Z}$, it is easily checked that y_k solves (2.8). If sectors $\mathcal{S}_k$ are defined as

$$\mathcal{S}_k : \qquad |\arg x - \frac{2k\pi}{n(M+1)}| < \frac{\pi}{n(M+1)} \ , \tag{3.3}$$

then as $x \to \infty$ in $\mathcal{S}_{k-\frac{n}{2}} \cup \mathcal{S}_{k-\frac{n}{2}+1} \cup \ldots \cup \mathcal{S}_{k+\frac{n}{2}}$, (3.1) implies that

$$\frac{d^p y_k}{dx^p} \sim (-1)^p \omega^{-k(1-n+2p)(M+1)/2} \frac{x^{(1-n+2p)M/2}}{i^{(n-1)/2}\sqrt{n}} \exp(-\omega^{-k(M+1)} x^{M+1}/(M+1)) \ . \tag{3.4}$$

We call *subdominant* a solution which tends to zero fastest in a given sector; up to a constant multiple, y_k is the unique solution to (2.8) which is subdominant in $\mathcal{S}_k$.

Wronskians between these solutions will be important in the following, so we define

$$W^{(m)}_{k_1, k_2, \ldots k_m} = W^{(m)}[y_{k_1}, y_{k_2}, \ldots y_{k_m}], \tag{3.5}$$

where

$$W^{(m)}[f_1, f_2, \ldots f_m] = \text{Det} \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \vdots & \vdots & & \vdots \\ f_1^{[m-1]} & f_2^{[m-1]} & \cdots & f_m^{[m-1]} \end{bmatrix} \tag{3.6}$$

and $f_i^{[p]}(x) = \frac{d^p}{dx^p} f_i(x)$. We will give a special status to those Wronskians whose arguments are successive integers, and in preparation for this we set

$$W^{(m)}_k = W^{(m)}_{k,k+1,k+2,\ldots k+m-1} . \tag{3.7}$$

---

*Note that since $\omega^{n(M+1)} = 1$, the shift in E in the definition of y_k could equally have been written as ω^{nk}, so long as k remains an integer. This would have been more in line with [1,4,11], and emphasises the fact that in the 'semiclassical' limit $M \to \infty$, the shifts tend to 1. Here we retain the conventions of [5], since this makes the transition to BA equations in standard form a little simpler, and also appears to be the most natural choice when half integer values of k come to be discussed, in §7 below.

Note that $W_k^{(1)} = y_k$, and that (generalising (3.2))

$$W_{k_1+k,k_2+k,\ldots k_m+k}^{(m)}(x, E, \mathbf{g}) = \omega^{m(n-m)k/2} W_{k_1,k_2,\ldots k_m}^{(m)}(\omega^{-k}x, \omega^{-nMk}E, \mathbf{g}) \ . \qquad (3.8)$$

For an nth order ODE with vanishing $(n-1)^{\text{th}}$ order term, as is true of our case by (2.6), it is standard that all n-fold Wronskians $W^{(n)}[f_1, \ldots f_n]$ are independent of x, and vanish if and only if the solutions $f_1, \ldots f_n$ are linearly dependent. In particular, (3.4) can be used as $|x| \to \infty$ in $\mathcal{S}_{k-1/2} \cup \mathcal{S}_{k+1/2}$ to show that $W_k^{(n)} = 1$, and so, for each $k \in \mathbb{Z}$, the functions $\{y_k, y_{k+1}, \ldots, y_{k+n-1}\}$ are linearly independent, and furnish a basis of solutions to (2.8).

4 Stokes multipliers and functional relations

We next establish a basic functional relation. Expanding y_0 in the basis provided by $\{y_1, y_2, \ldots, y_n\}$, it must be possible to write

$$\sum_{k=0}^{n} (-1)^k C^{(k)}(E, \mathbf{g}) \, y_k(x, E) = 0 \ , \qquad (4.1)$$

where $C^{(0)}(E, \mathbf{g}) = 1$ and the Stokes multipliers $C^{(k)}(E, \mathbf{g})$, $k > 0$ are analytic functions of E and $\mathbf{g}$. (The formal similarity between this equation and equation (5.50) of [13] should be noted.) Given that $W_1^{(n)} = 1$, we have the simple relation

$$C^{(k)}(E, \mathbf{g}) = W_{01\ldots\hat{k}\ldots n}^{(n)}(E, \mathbf{g}) \qquad (4.2)$$

with the hat $(\hat{k})$ indicating that the corresponding index is to be omitted. Clearly, $C^{(n)} = W_0^{(n)} = 1$. To treat the other Stokes multipliers, we first relate them to the 'privileged' Wronskians $W_k^{(m)}$. We will only need $C^{(1)}$ here, but the discussion can be generalised to the other cases. For general p, consider the determinants

$$0 = \text{Det} \begin{bmatrix} y_0^{[i]} & y_1^{[i]} & \cdots & y_p^{[i]} \\ y_0 & y_1 & \cdots & y_p \\ y_0' & y_1' & \cdots & y_p' \\ \vdots & \vdots & & \vdots \\ y_0^{[p-1]} & y_1^{[p-1]} & \cdots & y_p^{[p-1]} \end{bmatrix} \qquad (4.3)$$

with $i = 0, 1, \ldots p-2$. Expanding the first row of each using Cramer's rule, we have

$$0 = \sum_{k=0}^{p} (-1)^k \begin{bmatrix} y_k \\ y_k' \\ \vdots \\ y_k^{[p-2]} \end{bmatrix} W_{01\ldots\hat{k}\ldots p-1}^{(p)} \ . \qquad (4.4)$$

4

Now form a $(p-1) \times (p-1)$ matrix with the RHS of (4.4) as the first column, and the vectors $\mathbf{v}_j = [y_j, y'_j, \ldots y_j^{[p-2]}]^t$, $j = 2, 3 \ldots p-1$ as the remainder. Taking its determinant yields the following Plucker-type relation:

$$0 = W^{(p-1)}_{02\ldots p-1} W^{(p)}_{12\ldots p} - W^{(p-1)}_{12\ldots p-1} W^{(p)}_{02\ldots p} + (-1)^p W^{(p-1)}_{p2\ldots p-1} W^{(p)}_{01\ldots p-1} \, . \qquad (4.5)$$

Rearranging, and supplementing the notation (3.7) with the convention $W^{(0)}_k = 1 \; \forall k$,

$$\frac{W^{(p)}_{02\ldots p}}{W^{(p)}_1} \;=\; \frac{W^{(p-1)}_{02\ldots p-1}}{W^{(p-1)}_1} + \frac{W^{(p-1)}_2}{W^{(p-1)}_1} \frac{W^{(p)}_0}{W^{(p)}_1} \;=\; \sum_{m=0}^{p-1} \frac{W^{(m)}_2}{W^{(m)}_1} \frac{W^{(m+1)}_0}{W^{(m+1)}_1} \, , \qquad (4.6)$$

the second equality being obtained by repeated substitution. This is the result we need. Since $C^{(1)} = W^{(n)}_{02\ldots n}$, we can set $p = n$ and multiply (4.6) through by $\prod_{j=0}^n W^{(j)}_1$ to find

$$C^{(1)} \prod_{j=0}^n W^{(j)}_1 \;=\; \sum_{m=0}^{n-1} \left(\prod_{j=0}^{m-1} W^{(j)}_1 \right) W^{(m)}_2 W^{(m+1)}_0 \left(\prod_{j=m+2}^n W^{(j)}_1 \right) . \qquad (4.7)$$

5   Bethe ansatz equations

As it stands, equation (4.7) involves x as well as E and $\mathbf{g}$, and so is not quite ready to be mapped onto a standard set of BA equations. The simplest idea, simply to set x to zero, only works at exceptional values of the g_i, since in general the differential operator $D(\mathbf{g})$ has a (regular) singularity at $x = 0$. Instead, motivated by the treatments of [2, 4, 5], we expand $y(x, E, \mathbf{g})$ as

$$y(x, E, \mathbf{g}) = W^{(1)}_0(x, E, \mathbf{g}) = \sum_{i=0}^{n-1} D^{(1)}_{[i]}(E, \mathbf{g}) \, \chi_i(x, E, \mathbf{g}) \qquad (5.1)$$

where the χ_i form an alternative basis of solutions to (2.1), fixed by the demand that they have the simplest possible behaviours near the origin:

$$\chi_i \sim x^{g_i} + O(x^{g_i+n}), \qquad x \to 0 \, . \qquad (5.2)$$

(Recall that the g_i are the roots of the indicial polynomial (2.7).) Strictly speaking the asymptotic (5.2) does not always suffice to pin down *all* of the χ_i. Assume, until further notice, that the g_i's are ordered as

$$\Re e(g_0) < \Re e(g_1) < \ldots < \Re e(g_{n-1}) \, . \qquad (5.3)$$

Then χ_0 is certainly uniquely determined by (5.2), and the $\chi_{i>0}$ can be defined by a process of analytic continuation from this solution, just as is done for the radially-symmetric

Schrödinger equation when passing between the so-called regular and irregular solutions. This will be used in §8 below.

The direct substitution of (5.1) into (3.5) yields an expansion for $W_0^{(m)}(x, E, \mathbf{g})$ in which the functions $\chi_i(\omega^{-k}x, \omega^{-nMk}E, \mathbf{g})$ appear as well as the $\chi_i(x, E, \mathbf{g})$. However, all of these functions are solutions to the initial ODE, and by considering their behaviour near $x = 0$ one finds

$$\chi_i(\omega^{-k}x, \omega^{-nMk}E, \mathbf{g}) = \omega^{-kg_i}\chi_i(x, E, \mathbf{g}) \,. \tag{5.4}$$

Therefore the m-fold Wronskians $W_0^{(m)}$ have expansions of the form

$$W_0^{(m)}(x, E, \mathbf{g}) = \sum_{0 \leq j_1 < j_2 < \ldots < j_m \leq n-1} D_{[j_1 j_2 \ldots j_m]}^{(m)}(E, \mathbf{g}) \, W^{(m)}[\chi_{j_1}, \chi_{j_2} \ldots \chi_{j_m}](x, E, \mathbf{g}) \,. \tag{5.5}$$

Using (5.4), (5.1) and (3.5), the coefficients $D_{[j_1 j_2 \ldots j_m]}^{(m)}(E, \mathbf{g})$ can be expressed as sums of products of the $D_{[i]}^{(1)}(\omega^{-nMk}E, \mathbf{g})$. This leads to relations which generalise the 'quantum Wronskians' of [11]. However, for current purposes it is better to treat the coefficients with different values of m as independent functions, and so we will leave further discussion of this point to another occasion.

We will initially focus on the dominant terms of the expansions (5.5). With the ordering (5.3), these are

$$W_0^{(m)}(x, E, \mathbf{g}) \sim D_{[01 \ldots m-1]}^{(m)}(E, \mathbf{g}) \, x^{\beta_m + m(n-m)/2} \,, \qquad x \to 0 \tag{5.6}$$

where in order to simplify subsequent calculations we set

$$\beta_m = \sum_{j=0}^{m-1} g_j - m(n-1)/2 \,. \tag{5.7}$$

Substituting (5.5) into (4.7), an x-independent equation is found by extracting the coefficient of the leading power x^α, $\alpha = \sum_{j=0}^{n}(\beta_j + j(n-j)/2)$. Shifting E to $\omega^{nM}E$ and setting

$$D^{(m)}(E, \mathbf{g}) = D_{[01 \ldots m-1]}^{(m)}(E, \mathbf{g}) \,, \quad D_k^{(m)}(E, \mathbf{g}) = D^{(m)}(\omega^{-nMk}E, \mathbf{g}) \,,$$

$$T^{(1)}(E, \mathbf{g}) = C^{(1)}(\omega^{nM}E, \mathbf{g}) \,, \tag{5.8}$$

the final result can be written as

$$T^{(1)} \prod_{j=0}^{n} D_0^{(j)} = \sum_{m=0}^{n-1} \left(\prod_{j=0}^{m-1} D_0^{(j)} \right) \omega^{-\beta_m} D_1^{(m)} \, \omega^{\beta_{m+1}} D_{-1}^{(m+1)} \left(\prod_{j=m+2}^{n} D_0^{(j)} \right) \,. \tag{5.9}$$

This is a generalised T-Q relation, with the $D^{(j)}$ playing the role of the eigenvalues of the Q (or A in [11]) operators. In the language of integrable lattice models relations

of this type are known as the dressed vacuum form for the transfer matrix eigenvalues (see for example [14, 15]).

We can now derive an initial set of BA equations. Suppose that the zeroes of $D^{(m)}(E)$ are at $E = F_k^{(m)}$, $k = 1, 2 \ldots \infty$. (For the next few equations we will leave the dependence of all functions on $\mathbf{g}$ implicit.) Setting $E = F_k^{(m)}$ in (5.9), the LHS vanishes, as do all but two terms in the sum on the RHS. Assuming, as will generically be the case, that there are no further vanishings, this gives us the following set of coupled equations for the $\{F_k^{(m)}\}$:

$$\frac{D^{(m-1)}(\omega^{-nM}F_k^{(m)})}{D^{(m-1)}(F_k^{(m)})} \frac{D^{(m)}(\omega^{nM}F_k^{(m)})}{D^{(m)}(\omega^{-nM}F_k^{(m)})} \frac{D^{(m+1)}(F_k^{(m)})}{D^{(m+1)}(\omega^{nM}F_k^{(m)})} = -\omega^{-2\beta_m+\beta_{m-1}+\beta_{m+1}}.$$
(5.10)

These form a system of Bethe Ansatz equations (BAE) of $SU(n)$ type (see, for example, equation (3.32) of [13]). However the reality properties are not very transparent when the equations are given in this form, and it is useful to make one more definition, setting

$$D^{(m)}(E, \mathbf{g}) = A^{(m)}(\omega^{-nM(m-1)/2}E, \mathbf{g}) , \quad F_k^{(m)} = \omega^{nM(m-1)/2}E_k^{(m)} .$$
(5.11)

Then the BAE (5.10) become

$$\frac{A^{(m-1)}(\omega^{-nM/2}E_k^{(m)})}{A^{(m-1)}(\omega^{nM/2}E_k^{(m)})} \frac{A^{(m)}(\omega^{nM}E_k^{(m)})}{A^{(m)}(\omega^{-nM}E_k^{(m)})} \frac{A^{(m+1)}(\omega^{-nM/2}E_k^{(m)})}{A^{(m+1)}(\omega^{nM/2}E_k^{(m)})} = -\omega^{-2\beta_m+\beta_{m-1}+\beta_{m+1}}.$$
(5.12)

These can also be written using the Cartan matrix C_{mt} of the $SU(n)$ Dynkin diagram:

$$\prod_{t=1}^{n-1} \omega^{C_{mt}\beta_t} \frac{A^{(t)}(\omega^{\frac{nM}{2}C_{mt}}E_k^{(m)})}{A^{(t)}(\omega^{-\frac{nM}{2}C_{mt}}E_k^{(m)})} = -1, \qquad k = 1, 2, \ldots .$$
(5.13)

Finally, the BAE can be given a factorised form once the growth properties of the functions involved have been established. A WKB treatment along the lines of [4] shows that the function $A^{(1)}(E, \mathbf{g}) = D_{[0]}^{(1)}(E, \mathbf{g})$ has the large $|E|$ asymptotic

$$\ln A^{(1)}(E, \mathbf{g}) \sim \kappa(nM, n)(-E)^\mu, \qquad |E| \to \infty , \ |\arg(-E)| < \pi ,$$
(5.14)

where

$$\mu = \frac{(M+1)}{nM} , \quad \kappa(a, b) = \frac{\Gamma(1+\frac{1}{a})\Gamma(1+\frac{1}{b})\sin(\frac{\pi}{b})}{\Gamma(1+\frac{1}{a}+\frac{1}{b})\sin(\frac{\pi}{b}+\frac{\pi}{a})} .$$
(5.15)

The asymptotics for the remaining $\ln A^{(m)}(E, \mathbf{g})$ are more tricky. So long as it is assumed that the other functions $D_{[i]}^{(1)}$ share the asymptotic (5.14), then the quantum Wronskian-like relations mentioned just after (5.5) can be used to show that all of the $A^{(m)}(E, \mathbf{g})$ are of the same order, namely μ. For this section this suffices, but for the NLIE a more

7

precise result will be required. Unfortunately, cancellations of leading terms seem to be at work inside the quantum Wronskians in almost all directions in the complex E plane, making a direct calculation rather difficult. However, indirect evidence suggests the following:

$$\ln A^{(m)}(E, \mathbf{g}) \sim \frac{\sin(\pi m/n)}{\sin(\pi/n)} \kappa(nM, n)(-E)^\mu \quad , \quad |\arg(-E)| < \pi \ . \tag{5.16}$$

Note that this claim matches the $\mathbb{Z}_2$ symmetry property of the spectral determinants discussed in §7 below, and its implications are in agreement with the numerical checks for the "soluble" $M = 1/n$ cases reported in §8.

For $M > 1/(n{-}1)$ the order μ of the functions $A^{(m)}(E, \mathbf{g})$ is less than one, so the Hadamard factorization theorem implies

$$A^{(m)}(E, \mathbf{g}) = A^{(m)}(0, \mathbf{g}) \prod_{j=1}^{\infty} \left(1 - \frac{E}{E_j^{(m)}}\right) \ . \tag{5.17}$$

This finally allows (5.13) to be written as

$$\prod_{t=1}^{n-1} \omega^{C_{mt}\beta_t} \prod_{j=1}^{\infty} \left(\frac{E_j^{(t)} - \omega^{\frac{nM}{2}C_{mt}} E_k^{(m)}}{E_j^{(t)} - \omega^{-\frac{nM}{2}C_{mt}} E_k^{(m)}} \right) = -1 \,, \qquad k = 1, 2, \dots \ . \tag{5.18}$$

6 The nonlinear integral equation

Next we derive a set of coupled nonlinear integral equations. Define

$$a^{(m)}(E, \mathbf{g}) = \prod_{t=1}^{n-1} \omega^{-C_{mt}\beta_t} \frac{A^{(t)}\left(\omega^{-\frac{nM}{2}C_{mt}} E, \mathbf{g}\right)}{A^{(t)}\left(\omega^{\frac{nM}{2}C_{mt}} E, \mathbf{g}\right)} \tag{6.1}$$

so that $a^{(m)}(E_k^{(m)}, \mathbf{g}) = -1$ by (5.13). We will follow the ideas of [11, 16] (another approach can be found in [17]). The product representation (5.17) allows $\ln a^{(m)}$ to be written as an infinite sum over the zeroes $E_k^{(m)}$ of $A^{(m)}$. We now make two important assumptions: that all of the $E_k^{(m)}$ lie on the positive real axis of the complex E plane, and that these are the only points in some narrow strip about this axis at which $a^{(m)}(E, \mathbf{g})$ is equal to -1. We expect that these will hold for some range of the parameters $\mathbf{g}$. Cauchy's theorem can then be used to replace the sum by an integral along a contour $\mathcal{C}$ which runs from $+\infty$ to 0 above the real axis, encircles the origin and returns to ∞ below the real axis:

$$\ln a^{(m)}(E, \mathbf{g}) = \frac{-2\pi i}{n(M{+}1)} \sum_{t=1}^{n-1} C_{mt}\,\beta_t + \sum_{t=1}^{n-1} \int_{\mathcal{C}} \frac{dE'}{2\pi i} F_{mt}(E/E') \partial_{E'} \ln(1{+}a^{(t)}(E', \mathbf{g})) \tag{6.2}$$

8

where

$$F_{mt}(E) = \ln \frac{1 - \omega^{-\frac{nM}{2}C_{mt}} E}{1 - \omega^{\frac{nM}{2}C_{mt}} E} \quad . \tag{6.3}$$

After a variable change $E = \exp(\theta/\mu)$, we define (with a mild abuse of notation) $\ln a^{(m)}(\theta) \equiv \ln a^{(m)}(e^{\theta/\mu}, \mathbf{g})$, integrate by parts and use the property $[a^{(m)}(\theta)]^* = a^{(m)}(\theta^*)^{-1}$ to find

$$\ln a^{(m)}(\theta) - \sum_{t=1}^{n-1} \int_{-\infty}^{\infty} d\theta' R_{mt}(\theta - \theta') \ln a^{(t)}(\theta') = \frac{-2\pi i}{n(M+1)} \sum_{t=1}^{n-1} C_{mt}\,\beta_t \tag{6.4}$$

$$- 2i \sum_{t=1}^{n-1} \int_{-\infty}^{\infty} d\theta' R_{mt}(\theta - \theta') \Im m \ln(1 + a^{(t)}(\theta' - i0)) \ .$$

Here $R_{mt}(\theta) = \frac{i}{2\pi} \partial_\theta F_{mt}(e^{\theta/\mu})$. We now take the Fourier transform of this equation, setting

$$\widetilde{f}(k) = \mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(\theta)\, e^{-i\theta k}\, d\theta \ , \tag{6.5}$$

$$f(\theta) = \mathcal{F}^{-1}[\widetilde{f}](\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{f}(k)\, e^{i\theta k}\, dk \ . \tag{6.6}$$

The transformed equations can be written in a compact form as

$$\sum_{t=1}^{n-1} (\delta_{mt} - \widetilde{R}_{mt}(k)) \mathcal{F}\Big[\ln a^{(t)}\Big](k) = -\frac{4i\pi^2 \delta(k)}{n(M+1)} \sum_{t=1}^{n-1} C_{mt}\,\beta_t$$

$$- 2i \sum_{t=1}^{n-1} \widetilde{R}_{mt}(k) \Im m \mathcal{F}\Big[\ln(1 + a^{(t)})\Big](k) \ , \tag{6.7}$$

where the non-vanishing entries of $\widetilde{R}_{mt}(k)$ are

$$\widetilde{R}_{<mt>}(k) = \frac{\sinh(\frac{\pi}{n}\xi k)}{\sinh(\frac{\pi}{n}(1 + \xi)k)} \ , \quad \widetilde{R}_{mm}(k) = \frac{\sinh(\frac{\pi}{n}(1 - \xi)k)}{\sinh(\frac{\pi}{n}(1 + \xi)k)} \ , \quad (\xi = \frac{1}{M}) \tag{6.8}$$

the notation $< mt >$ implying that the nodes m and t are connected on the Dynkin diagram of $SU(n)$. Applying the inverse matrix $(\mathbb{1} - \widetilde{R}(k))^{-1}$, transforming back and rewriting the imaginary parts in terms of values above and below the real axis, we obtain a set of coupled NLIEs for the functions $a^{(m)}(\theta)$, $m = 1, \ldots, n-1$, considered along contours $\mathcal{C}_1$ and $\mathcal{C}_2$ which run from $-\infty$ to $+\infty$, just below and just above the real θ-axis:

$$\ln a^{(m)}(\theta) = i\pi\alpha_m(\mathbf{g}) - ib_0 M_m\, e^{\theta}$$

$$+ \sum_{t=1}^{n-1} \left(\int_{\mathcal{C}_1} d\theta' \varphi_{mt}(\theta - \theta') \ln(1 + a^{(t)}(\theta')) - \int_{\mathcal{C}_2} d\theta' \varphi_{mt}(\theta - \theta') \ln(1 + \frac{1}{a^{(t)}(\theta')}) \right). \tag{6.9}$$

9

The driving terms $ib_0 M_n e^\theta$ arise from zero modes, which can be traced to the poles of $(\mathbb{1} - \widetilde{R}(k))^{-1}$ at $k = i$, and their magnitudes

$$b_0 = 2\sin(\pi\mu)\kappa(nM, n) , \qquad M_m = \frac{\sin(\pi m/n)}{\sin(\pi/n)} \tag{6.10}$$

are fixed by imposing the asymptotic (5.16). The kernel and twist factors are, respectively,

$$\varphi_{mt}(\theta) = \mathcal{F}^{-1}\left[\left(\mathbb{1} - (\mathbb{1} - \widetilde{R}(k))^{-1}\right)_{mt}\right](\theta) , \tag{6.11}$$

$$\alpha_r(\mathbf{g}) = -\frac{2\pi}{n(M+1)} \sum_{t,m=1}^{n-1} \left(\mathbb{1} - \widetilde{R}(0)\right)^{-1}_{rm} C_{mt}\, \beta_t . \tag{6.12}$$

These can be written more explicitly with the help of the deformed Cartan matrix

$$C_{mt}(k) = \begin{cases} 2 & m = t, \\ \frac{-1}{\cosh(\frac{\pi k}{n})} & < mt > \end{cases} \tag{6.13}$$

and its inverse

$$C_{tm}^{-1}(k) = C_{mt}^{-1}(k) = \frac{\coth(\frac{\pi}{n}k)\sinh(\frac{\pi}{n}(n-m)k)\sinh(\frac{\pi}{n}t\,k)}{\sinh(\pi k)} \qquad (m \geq t) . \tag{6.14}$$

By (6.8), the non-vanishing off-diagonal and diagonal elements of $\widetilde{R}(k)$ are related by $2\widetilde{R}_{<mt>}(k) = (1 - \widetilde{R}_{mm}(k))/\cosh(\frac{\pi k}{n})$, and so

$$\delta_{mt} - \widetilde{R}_{mt}(k) = \frac{1}{2}C_{mt}(k)(1 - \widetilde{R}_{mm}(k)) , \tag{6.15}$$

where

$$1 - \widetilde{R}_{mm}(k) = \frac{2\sinh(\frac{\pi}{n}\xi k)\cosh(\frac{\pi}{n}k)}{\sinh(\frac{\pi}{n}(1+\xi)k)} . \tag{6.16}$$

Plugging relations (6.15) and (6.16) into (6.12) and (6.11) we obtain

$$\varphi_{mt}(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik\theta}\left(\delta_{mt} - \frac{\sinh(\frac{\pi}{n}(1+\xi)k)}{\sinh(\frac{\pi}{n}\xi k)\cosh(\frac{\pi}{n}k)}C_{mt}^{-1}(k)\right) ,$$

$$\alpha_m(\mathbf{g}) = -\frac{2}{n}\beta_m , \tag{6.17}$$

with $\xi = 1/M$. The final NLIEs exactly match the massless versions of the equations found in [18, 19] in a completely different context.

7 Symmetry properties

The $SU(n)$ Dynkin diagram has an evident $\mathbb{Z}_2$ symmetry, swapping nodes m and $n-m$. This symmetry is also respected by the BAE so long as the twists are transformed at the same time. We would expect to see signs of this symmetry in the differential equation, but quite how is not immediately obvious. For example, the first node of the Dynkin diagram is associated with the basic solution $y(x, E, \mathbf{g})$ while its partner, the $(n-1)^{\text{th}}$ node, is associated with an $(n-1)$-fold Wronskian $W_0^{(n-1)}(x, E, \mathbf{g})$. The answer to this puzzle is provided by the fact that taking the Wronskian of $n-1$ solutions to an nth order ODE automatically provides a solution to the adjoint equation. For $n=3$ this observation can be found in [20], and was related to the current context in [5]. For general n, the result can be established as follows. It is easiest not to use the factorised form (2.5) of $D(\mathbf{g})$ for the differential operator, but rather to suppose that (2.8) has been rewritten as

$$\frac{d^n}{dx^n}\psi + B_2(x)\frac{d^{n-2}}{dx^{n-2}}\psi + B_3(x)\frac{d^{n-3}}{dx^{n-3}}\psi + \ldots + (-1)^{n+1}P(x, E)\psi = 0. \qquad (7.1)$$

(For the differential operators discussed in §2, $B_k(x) = (-1)^{n+1}A_k x^{-k}$, but neither this nor the explicit relation between the A_k's and the g_i's will be needed for the argument.) Now consider $n-1$ solutions to this equation, gathered together into a vector as $\mathbf{f} = (f_1, f_2, \ldots f_{n-1})$. We denote by $V[a_1, a_2, \ldots a_{n-1}]$ the determinant of the matrix whose rows are the derivatives $\mathbf{f}^{[a_1]}, \mathbf{f}^{[a_1]}, \ldots \mathbf{f}^{[a_{n-1}]}$, so that the Wronskian of $f_1, f_2 \ldots f_{n-1}$ is

$$W^{(n-1)}[f_1, f_2, \ldots f_{n-1}](x, E, \mathbf{g}) = V[0, 1, \ldots, n-2]. \qquad (7.2)$$

Differentiating once, and using the general property that the differential of a determinant is equal to a sum of determinants in which each row has been differentiated,

$$\frac{d}{dx}V[a_1, a_2, \ldots a_{n-1}] = \sum_{k=1}^{n-1}V[a_1, \ldots a_{k-1}, a_k+1, a_{k+1}, \ldots a_{n-1}], \qquad (7.3)$$

we have

$$\frac{d}{dx}W^{(n-1)} = V[0, 1, \ldots n-4, n-3, n-1], \qquad (7.4)$$

all other terms vanishing by antisymmetry. Differentiating again,

$$\frac{d^2}{dx^2}W^{(n-1)} = V[0, 1, \ldots n-4, n-2, n-1] + V[0, 1, \ldots n-4, n-3, n]. \qquad (7.5)$$

The last term can be rewritten using (7.1) to substitute for $\mathbf{f}^{[n]}$:

$$V[0, 1, \ldots n-4, n-3, n] = -B_2(x)V[0, 1, \ldots n-4, n-3, n-2] = -B_2(x)W^{(n-1)}. \qquad (7.6)$$

(Again, all other terms vanish by antisymmetry.) Thus

$$\left(\frac{d^2}{dx^2} + B_2(x)\right)W^{(n-1)} = V[0, 1, \ldots n-4, n-2, n-1]. \qquad (7.7)$$

11

The pattern should now be clear. Differentiating both sides of (7.7) will again result in two terms on the RHS. The second of these can be rewritten using (7.1), but this time yields $B_3(x)W^{(n-1)}$ instead of $-B_2(x)W^{(n-1)}$ as the one non-vanishing contribution. Taking this over onto the LHS and continuing to differentiate, one finally finds that $W^{(n-1)}$ satisfies

$$\left(\frac{d^n}{dx^n} + \frac{d^{n-2}}{dx^{n-2}}B_2(x) - \frac{d^{n-3}}{dx^{n-3}}B_3(x) + \ldots - P(x,E)\right)W^{(n-1)} = 0, \qquad (7.8)$$

which is indeed (up to an overall sign) the equation adjoint to (7.1). Now we can return to the factorised form of $D(\mathbf{g})$, take its adjoint using (2.3), and rewrite (7.8) as:

$$\left((-1)^{n+1}D(\mathbf{g}^\dagger) + (-1)^n P(x,E)\right)W^{(n-1)}(x,E,\mathbf{g}) = 0, \quad P(x,E) = x^{nM} - E, \quad (7.9)$$

with

$$\mathbf{g}^\dagger = (g_0^\dagger, g_1^\dagger, \ldots g_{n-1}^\dagger), \qquad g_j^\dagger = n-1-g_{n-1-j}. \qquad (7.10)$$

For n even this is the original ODE (2.8), modulo the swap to the 'conjugate' set of parameters $\mathbf{g}^\dagger$. For n odd, the term $P(x,E)$ has been replaced by $-P(x,E)$. However, the conventions adopted in the definition (3.2) mean that the 'half-shifted' functions $y_{k+1/2}(x,E,\mathbf{g})$ ($k \in \mathbb{Z}$) themselves solve (2.8) with P replaced by $-P$. Taking an $(n-1)$-fold Wronskian of these functions and repeating the above discussion, the minus signs cancel. In particular, this means that whether n is even or odd we have

$$\left((-1)^{n+1}D(\mathbf{g}^\dagger) + P(x,E)\right)W^{(n-1)}_{1-n/2}(x,E,\mathbf{g}) = 0. \qquad (7.11)$$

This is exactly the equation solved by the $y_k(x,E,\mathbf{g}^\dagger)$. Comparing asymptotics as $|x| \to \infty$ in $\mathcal{S}_0$, we finally establish the formula

$$W^{(n-1)}_{1-n/2}(x,E,\mathbf{g}) = y(x,E,\mathbf{g}^\dagger). \qquad (7.12)$$

This shows that the $(n-1)$-fold Wronskian is (up to a shift in its arguments) just a basic solution to another ODE, and thus demonstrates that the relation between the first and last nodes is indeed reflected in the properties of the differential equation. The spectral functions corresponding to the remaining nodes on the Dynkin diagram can now be obtained either as m-fold Wronskians of the $W^{(n-1)}_{k+1/2}$, or as $(n-m)$-fold Wronskians of the y_k, and the diagram symmetry should be reflected in the equality of the results of the two calculations. Keeping the normalisation of the y's as in (3.4), the necessary identity is the following:

$$W^{(n-m)}_{m-1} = W^{(m)}[W_0^{(n-1)}, W_1^{(n-1)}, \ldots, W_{m-1}^{(n-1)}]. \qquad (7.13)$$

While we do not have a general proof, for $m=2$ this result follows from the Jacobi identity (cf. for example equation (2.20) of [13]) and the fact that $W^{(n)} = 1$. This case is also equivalent to a formula due to Frobenius [21], more recently reviewed in [22].

8 Analytic continuation and the linear potential

Thus far, we have restricted attention to the dominant terms in the expansions (5.5). However, quite which term is dominant depends on the ordering (5.3) of the g_i's. While the ODE (2.8) is insensitive to this ordering (as remarked earlier, it depends on the g_i in a symmetrical manner), the same is not true of the NLIE (6.9), since it sees not the parameters $\mathbf{g}$ but rather the BA twists $\alpha(\mathbf{g})$. By analytic continuation, it should be possible to move between different twists corresponding to the same ODE, thereby accessing the other terms in the expansions. In this section we will test this idea by means of a simple example. This also serves as a rather strong check on the various conjectures that have been made above.

In the expansion of y given by (5.1), the coefficients $D_{[i]}^{(1)}$ are explicitly

$$D_{[i]}^{(1)}(E, \mathbf{g}) = (-1)^i \frac{W^{(n)}[W_0^{(1)}(x, E, \mathbf{g}), \chi_0, \cdots \chi_{i-1}, \chi_{i+1}, \cdots \chi_{n-1}]}{W^{(n)}[\chi_0, \cdots \chi_{n-1}]} \tag{8.1}$$

where

$$W^{(n)}[\chi_0, \cdots \chi_{n-1}] = \prod_{j=0, i=j+1}^{n-1} (g_j - g_i) . \tag{8.2}$$

Thus, for example, continuation of the parameters from $\mathbf{g} = (g_0, \ldots g_i, \ldots g_j, \ldots g_{n-1})$ to $\mathbf{g}' = (g_0, \ldots g_j, \ldots g_i, \ldots g_{n-1})$ has the following effect

$$D_{[i]}^{(1)}(E, \mathbf{g}') = D_{[j]}^{(1)}(E, \mathbf{g}) , \qquad D_{[j]}^{(1)}(E, \mathbf{g}') = D_{[i]}^{(1)}(E, \mathbf{g}) , \tag{8.3}$$

with the other $D_{[k]}^{(1)}(E, \mathbf{g})$ remaining unchanged. So long as the property (2.6) is preserved at all points of the continuation, the NLIE will hold throughout. At the end of the continuation, the ODE will have returned to its original form, but the NLIE will have undergone a nontrivial monodromy, since the twists will have changed. Explicitly, an NLIE describing $D_{[i]}^{(1)}(E, \mathbf{g})$ can be found by continuing the twist parameter α to α' via the transformation $g_0 \to g_i$, with the remaining g_k being allowed to permute amongst themselves. In fact, a number of different NLIEs can be obtained, depending on the permutation chosen.

A set of special choices for the vector $\mathbf{g}$ provides a simple illustration of all this. We use the notation $\hat{\mathbf{g}}$ to indicate a vector with distinct components g_j taking values in the set $\{0, 1, \ldots (n-1)\}$. There are $n!$ different $\hat{\mathbf{g}}$, but for each choice the resulting differential operator $D(\hat{\mathbf{g}})$ is equal to $\frac{d^n}{dx^n}$. We will use $\hat{\mathbf{g}}_i$ to indicate g_0 has been fixed to be equal to i. In this particularly simple case, the functions (8.1) become

$$D_{[i]}^{(1)}(E, \hat{\mathbf{g}}_i) = \frac{1}{i!} \frac{d^i}{dx^i} y(0, E, \hat{\mathbf{g}}) . \tag{8.4}$$

(Since the function $y(x, E, \hat{\mathbf{g}})$ does not depend on the ordering of the components of $\hat{\mathbf{g}}$, we omit the subscript i on the RHS of this equation.) The direct treatment of

13

previous sections only allowed us to discuss $D^{(1)}_{[0]} = y(0, E, \hat{\mathbf{g}})$, but according to the continuation idea just described, the NLIEs should also encode information about the higher derivatives $y^{[i]}$.

To make a numerical check, we set $M = 1/n$, so that $P(x, E) = x - E$ and the ODE is solvable by a complex Fourier transform, as in [5] for $n = 3$. This lies outside the range $M > 1/(n-1)$, but, just as in previous examples [1,5], we assume that the final NLIE continues to hold. If we write $y(x, E, \hat{\mathbf{g}}_i) = \mathcal{A}(x-E)$, then the ODE becomes

$$(-1)^{n+1} \frac{d^n \mathcal{A}(x)}{dx^n} + x\mathcal{A}(x) = 0 \tag{8.5}$$

and has the normalised solution

$$\mathcal{A}(x) = \frac{1}{i^{(n-1)/2}\sqrt{2\pi}} \int_\Gamma dp \ e^{-ipx + \frac{(ip)^{n+1}}{n+1}} \ . \tag{8.6}$$

The contour Γ follows a path in the lower half complex plane that starts at $|x| = \infty$ along the ray $\arg(x) = -\frac{\pi}{2} - \frac{\pi}{n+1}$ and ends along $\arg(x) = -\frac{\pi}{2} + \frac{\pi}{n+1}$ (for $n=3$ this is illustrated in figure 2 of [5]). Evaluating (8.6) numerically for $n = 3, 4$ and 5, we achieved good agreement with results obtained via the numerical treatment of (6.9), not only for the zeroes of the functions $y(0, E, \hat{\mathbf{g}})$ but also, after analytic continuation from $\mathbf{g}_0$ to $\mathbf{g}_1$, $\mathbf{g}_2$ and so on, of $y'(0, E, \hat{\mathbf{g}}), \ldots, y^{[n-1]}(0, E, \hat{\mathbf{g}})$. The accuracy obtained was typically to 12 digits, and could doubtless be improved with longer computing runs. Occasionally, the NLIE (6.9) breaks down, and in particular attempts to solve it by numerical iteration fail to converge. Generally this happens whenever there is a breakdown in one or other of the two assumptions necessary for the derivation of the NLIE in its simplest form, mentioned at the beginning of §6 above. For the continuations between the different $\hat{\mathbf{g}}_i$ it seems to be the second assumption which sometimes fails, with points appearing on the positive real axis of the complex E plane, not in the set $\{E^{(m)}_k\}$, at which the functions $a^{(m)}$ take the value -1. (For $n = 2$ these points are zeroes of $T(E)$, and the necessary modification of the NLIE was discussed in [23].) In most cases we found that by altering the choice of $\hat{\mathbf{g}}_i$ the problem could anyway be avoided. One exceptional case is $n=3$. The two choices for $\hat{\mathbf{g}}_1$ are $(1, 0, 2)$ and $(1, 2, 0)$, and for neither of these did iteration of the basic NLIE converge. Despite this fact, an extrapolation allowed a comparison with the results of the complex Fourier transform to be made, and we found that the zeroes of $y'(0, E, \hat{\mathbf{g}})$ were indeed reproduced. Note that in this case the zeroes of $y(0, E, \hat{\mathbf{g}})$ and $y''(0, E, \hat{\mathbf{g}})$ can be found from an NLIE which respects the $SU(3)$ diagram symmetry [5], but that to find the zeroes of $y'(0, E, \hat{\mathbf{g}})$, we are forced to use an integral equation which breaks this symmetry.

9 Duality

As in [2,4,5], a duality transformation can be defined which relates solutions of the ODEs with $M > 0$ with solutions of ODEs of the same form but with $-1 < M < 0$. For $n = 2$,

14

the case of the Schrödinger equation, this relates a confining potential to a singular potential. The first step is to implement a (generalised) Langer [24] transformation

$$z = \ln x \, , \quad y(x) = e^{(n-1)z/2} u(z) \, . \tag{9.1}$$

The use of the operators $D(g)$ and $D(\mathbf{g})$ makes this task very simple, since the relation

$$\left(\frac{d}{dx} - \frac{g}{x} \right) e^{\frac{p-1}{2} z(x)} f(z(x)) = e^{\frac{p-3}{2} z} \left(\frac{d}{dz} + \frac{p-1}{2} - g \right) f(z) \tag{9.2}$$

allows exponentials of z to be passed successively through the factors of $D(\mathbf{g})$, resulting in the transformed equation

$$\left[(-1)^{n+1} \left(\frac{d}{dz} - \gamma_{n-1} \right) \cdots \left(\frac{d}{dz} - \gamma_1 \right) \left(\frac{d}{dz} - \gamma_0 \right) + e^{n(1+M)z} - E e^{nz} \right] u(z) = 0 \tag{9.3}$$

where $\gamma_i = g_i - (n-1)/2$. The exponentials can now be exchanged via $z \to \frac{z}{M+1} + \ln \frac{M+1}{E^{1/n}}$ to obtain

$$\left[(-1)^{n+1} \left(\frac{d}{dz} - \frac{\gamma_{n-1}}{M+1} \right) \cdots \left(\frac{d}{dz} - \frac{\gamma_1}{M+1} \right) \left(\frac{d}{dz} - \frac{\gamma_0}{M+1} \right) - \tilde{E} e^{nz} - e^{\frac{nz}{M+1}} \right] u(z) = 0 \, , \tag{9.4}$$

where $\tilde{E} = -(M+1)^{nM} E^{-(M+1)}$. Applying the inverse transformation, (2.8) becomes

$$\left((-1)^{n+1} D(\widetilde{\mathbf{g}}) + P(x, \widetilde{E}) \right) y(x) = 0 \, , \qquad P(x, \widetilde{E}) = -x^{n\widetilde{M}} - \widetilde{E} \, , \tag{9.5}$$

with

$$\widetilde{M} = -\frac{M}{M+1} \, , \qquad \widetilde{\mathbf{g}} = \{ \widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_{n-1} \} \, , \qquad \widetilde{g}_i = \frac{g_i + M(n-1)/2}{M+1} \, . \tag{9.6}$$

(Observe that the property (2.6) is preserved by the transformation of $\mathbf{g}$.) This, the dual equation, is the same as the original ODE (2.8) apart from the reversed sign in front of the 'potential' term $x^{n\widetilde{M}}$, and the fact that the parameters $(M, \mathbf{g})$ have been replaced by $(\widetilde{M}, \widetilde{\mathbf{g}})$. Thus the equation transforms very simply under duality.

10 Conclusions

In this paper we have found a hidden $SU(n)$ structure inside certain n^{th}-order differential equations. This has enabled a large set of Bethe Ansatz systems, and their associated nonlinear integral equations, to be related directly to spectral problems.

One aspect that we did not discuss relates to the truncations of the functional relations which can occur at special values of the parameters M and $\mathbf{g}$. These can provide a link to equations arising in the thermodynamic Bethe ansatz, a subject of much independent interest. The earlier analysis of [6], which should be considered

as complementary to our work, was much closer to this approach, and it would be interesting to develop it further.

The correspondence between Bethe ansatz systems and ordinary differential equations has various potential applications. From the point of view of integrable systems, results such as duality properties are made much more apparent by the mapping into a differential equation, a fact which has already been exploited, for $n=2$, in [2]. In fact, at least for $n = 2$, the T and Q-functions are coming to have an increasingly important role to play in the general study of integrable quantum field theories with boundaries [25–30], and this makes it likely that the new perspective on these functions provided by their reinterpretation as spectral determinants will prove of broader relevance. (This subject is even starting to find applications in string theory [31].) To give one example, in the study of boundary flows direct relations between the T's and the g-functions of Affleck and Ludwig [32] have been found in certain cases [25, 26, 30].

On the other hand, it is also rather remarkable that one can obtain information about spectral determinants for quantum *mechanical* problems, and more general ODEs, using conformal field theory techniques such as the truncated conformal space approach [28] and perturbation theory [2, 26, 30]. Our treatment also entailed some rather strong conjectures about the reality and positivity properties of spectral problems associated with the ordinary differential equations under consideration. These conjectures were well-supported by numerical work, but proofs are still lacking. Previous examples of spectral problems which, despite having no obvious hermiticity, still exhibit real spectra include the $\mathcal{CP}$ symmetric quantum mechanics of [33]. (This was briefly discussed from the point of view of functional relations and T-Q systems in [4].) It appears that a much larger set of problems sharing similar properties is being uncovered, and it would be very rewarding to understand this behaviour at a deeper level.

Note added – After this paper was finished we were informed by J. Suzuki that he has also been working on this topic, with results which partially overlap those reported here. We also thank J. Suzuki for suggesting some improvements in our terminology.

References

[1] P. Dorey and R. Tateo, 'Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations', J. Phys. A32 (1999) L419–L425, `hep-th/9812211`

[2] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, 'Spectral determinants for Schrödinger equation and Q-operators of Conformal Field Theory', `hep-th/9812247`

[3] J. Suzuki, 'Anharmonic Oscillators, Spectral Determinant and Short Exact Sequence of $U_q(\hat{sl}_2)$', J. Phys. A32 (1999) L183–L188, `hep-th/9902053`

[4] P. Dorey and R. Tateo, 'On the relation between Stokes multipliers and the T-Q system of conformal field theory', Nucl. Phys. B563 (1999) 573–602, `hep-th/9906219`

[5] P. Dorey and R. Tateo, 'Differential equations and integrable models: the $SU(3)$ case', Nucl. Phys. B571 (2000) 583–606, `hep-th/9910102`

[6] J. Suzuki, 'Functional relations in Stokes multipliers and Solvable Models related to $U_q(A_n^{(1)})$', J. Phys. A33 (2000) 3507–3521, `hep-th/9910215`

[7] J. Suzuki, 'Functional relations in Stokes multipliers – Fun with $x^6 + \alpha x^2$ potential', `quant-ph/0003066`

[8] P.-F. Hsieh and Y. Sibuya, 'On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients', J. Math. Analysis and Applications 16 (1966) 84–103;
Y. Sibuya, *Global Theory of a second-order linear ordinary differential operator with polynomial coefficient* (Amsterdam: North-Holland 1975)

[9] R. Baxter, *Exactly solved models in statistical mechanics* (Academic Press 1982)

[10] A. Voros, 'Semi-classical correspondence and exact results: the case of the spectra of homogeneous Schrödinger operators', J. Physique Lett. 43 (1982) L1–L4;
— 'The return of the quartic oscillator. The complex WKB method', Ann. Inst. Henri Poincaré Vol XXXIX (1983) 211–338;
— 'Exact resolution method for general 1D polynomial Schrödinger equation', J. Phys. A32 (1999) 5993–6007, `math-ph/9903045`

[11] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, 'Integrable structure of conformal field theory II. Q-operator and DDV equation', Commun. Math. Phys. 190 (1997) 247–278, `hep-th/9604044`

[12] Earl A. Coddington and Norman Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill 1955

[13] I. Krichiver, O. Lipan, P. Wiegmann and A. Zabrodin, 'Quantum Integrable Models and Discrete Classical Hirota Equations', Commun. Math. Phys. 199 (1997) 267–304, `hep-th/9604080`

[14] N. Yu. Reshetikhin, 'The spectrum of the transfer matrices connected with Kac-Moody algebras', Lett. Math. Phys. 14 (1987) 235–246

[15] A. Kuniba and J. Suzuki, 'Analytic Bethe Ansatz for fundamental representations of Yangians', Commun. Math. Phys. 173 (1995) 225–264, `hep-th/9406180`

[16] C. Destri and H.J. de Vega, 'New thermodynamic Bethe ansatz equations without strings', Phys. Rev. Lett. 69 (1992) 2313–2317;
— 'Unified approach to thermodynamic Bethe ansatz and finite size corrections for lattice models and field theories', Nucl. Phys. B438 (1995) 413–454

[17] A. Klümper, M.T. Batchelor and P.A. Pearce, 'Central charges of the 6- and 19-vertex models with twisted boundary conditions', J. Phys. A24 (1991) 3111–3133

[18] A. Mariottini, 'Ansatz di Bethe termodinamico ed equazione di Destri-de Vega in teorie di campo bidimensionali', Degree thesis (in Italian), Bologna March 1996

[19] P. Zinn-Justin, 'Nonlinear integral equations for complex Affine Toda models associated to simply laced Lie algebras', J. Phys. A31 (1998) 6747–6770, `hep-th/9712222`

[20] G.D. Birkhoff, 'On the solutions of ordinary linear homogeneous differential equations of the third order', Ann. Math. 12 (1910-11) 103–127

[21] G. Frobenius, Crelle (Journal fur die reine und angewandte Mathematik) Vol 77 (1874), 248

[22] M. Bôcher, 'On Wronskians of functions of a real variable', Bull. AMS Vol VIII, October 1901, 53–63

[23] D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini, 'Excited state Destri de Vega equation for sine-Gordon and restricted sine-Gordon models', Phys. Lett. B390 (1997) 243–251, `hep-th/9608091`

[24] R.E. Langer, 'On the connection formulas and the solutions of the wave equation', Phys. Rev. 51 (1937) 669–676

[25] P. Fendley, F. Lesage and H. Saleur, ' Solving 1-D plasmas and 2-D boundary problems using Jack polynomials and functional relations', J. Stat. Phys. 79 (1995) 799, `hep-th/9409176`

[26] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, 'Integrable Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz', Commun. Math. Phys. 177 (1996) 381 –398, `hep-th/9412229`

[27] P. Fendley, F. Lesage and H. Saleur, 'A unified framework for the Kondo problem and for an impurity in a Luttinger liquid', J. Stat. Phys. 85 (1996) 211 –249, `cond-mat/9510055`

[28] P. Dorey, A. Pocklington, R. Tateo and G.M.T. Watts 'TBA and TCSA with boundaries and excited states', Nucl. Phys. B525 (1998) 641–663, `hep-th/9712197`

[29] F. Lesage, H. Saleur and P. Simonetti, 'Boundary flows in minimal models', Phys. Lett. B427 (1998) 85–92, `hep-th/9802061`

[30] P. Dorey, I. Runkel, R. Tateo and G.M.T. Watts, 'g function flow in perturbed boundary conformal field theories', Nucl. Phys. B578 (2000) 85 –122, `hep-th/9909216`

[31] J.A. Harvey, D. Kutasov and E.J. Martinec, 'On the relevance of tachyons', `hep-th/0003101`

[32] I. Affleck and A.W.W. Ludwig, 'Universal noninteger "Ground-State Degeneracy" in critical quantum systems', Phys. Rev. Lett. 67 (1991) 161–164

[33] C.M. Bender and S. Boettcher, 'Real spectra in non-hermitian Hamiltonians having $\mathcal{PT}$ symmetry', Phys. Rev. Lett. 80 (1998) 4243–5246, `physics/9712001`